Homework 1 SOLUTIONS: Monocentric City Model
Graduate Urban Economics, SUFE

In this homework I ask you to derive some of the basic results of the monocentric city model and answer a few questions. The setup is identical to that discussed in class (simple, linear city model):

1. Consumers have preferences over housing, q, and the numeraire good z: $U(q, z) = q^{\beta} z^{1-\beta}$
2. All consumers work in the CBD, earn wage w, and commute with per unit distance cost τ; a consumer at x has $\tau * x$ commuting cost
3. Let $P(x)$ and $R(x)$ stand for the housing price and land rent at distance x from the CBD
4. Land is owned by absentee landlords and rented to perfectly competitive developers. Developers have housing production function $H = K^{1-\alpha} * L^{\alpha}$; all land is developable. The cost of capital K is i and land L is rented at rate $R(x)$.
5. Simplest spatial structure: the city is represented by a line (one-dimensional, not two) with the CBD at the left-most endpoint

Problem 1: Derive the Alonso-Muth condition showing the gradient (slope) of the price function, $\frac{\partial P(x)}{\partial x}$ and explain the sign (is the gradient positive, negative, zero, or unknown).

Answer: We can derive the Alonso-Muth condition generally by totally differentiating the equal utility condition and then plugging in the optimization condition (MRS=price ratio). For Cobb-Douglas we have $U(q, z) = q^{\beta} z^{1-\beta}$ with budget constraint $w = \tau * x + P(x) * q(x) + z$. Optimizing using the FOC's gives the demand functions: 1) $q^*(x) = \frac{\beta(w-\tau*x)}{P(x)}$ and 2) $z^*(x) = w - \tau * x - P(x) * q^*(x) = (w - \tau * x)(1 - \beta)$. Then the indirect utility function is:

$$V(x) = \left[\frac{\beta(w - \tau * x)}{P(x)}\right]^{\beta} * [(w - \tau * x)(1 - \beta)]^{1-\beta} \tag{1}$$

In the spatial equilibrium households at all locations must have the same utility level $\bar{v}$. Setting the indirect utility to this level allows me to solve for $P(x)$ purely as a function of the parameters (explicit function). This is the bid-rent function, which shows the maximum price such that the household at x would attain utility $\bar{v}$:

$V(x) = \beta^{\beta} * (1 - \beta)^{\beta} * P(x)^{-\beta} * (w - \tau * x) = \bar{v}$ and thus $P(x) = \beta * (1 - \beta)^{\frac{1-\beta}{\beta}} \left[\frac{w-\tau*x}{\bar{v}}\right]^{\frac{1}{\beta}}$

However, to derive the Alonso-Muth condition it's better to write the indirect utility function leaving in $q^*(x)$, which I just denote as $q(x)$ for simplicity:

$$\begin{aligned} V(x) &= q(x)^{\beta} * (w - \tau * x - P(x) * q(x))^{1-\beta} = \bar{v} && (2) \\ P(x) &= \frac{w - \tau * x}{q(x)} - \left(\frac{\bar{v}}{q(x)}\right)^{\frac{1}{1-\beta}} && (3) \end{aligned}$$

Taking the partial derivative of the rental function with respect to location yields the Alonso-Muth condition:

$$\frac{\partial P(x)}{\partial x} = \frac{-\tau}{q(x)} \tag{4}$$

We can see that this expression must be negative and further, that it is convex. This is because $\partial q(x)/\partial x > 0$–as people live further from the CBD they increase their consumption of housing.

Problem 2: How does consumption of the numeraire change with distance from the CBD? Derive the gradient of numeraire consumption and give a brief interpretation (1-2 sentences is fine).
Answer: First take the derivative of numeraire consumption, $z = w - \tau * x - P(x) * q(x)$, with respect to x:

$$\frac{\partial z(x)}{\partial x} = -\tau - \left[\frac{\partial P(x)}{\partial x} * q(x) + P(x) * \frac{\partial q(x)}{\partial x}\right] \tag{5}$$

From the above equation the sign is not clear. However, by plugging in the Alonso-Muth condition for $\partial P(x)/\partial x$ we get:

$$\frac{\partial z(x)}{\partial x} = -P(x) * \frac{\partial q(x)}{\partial x} < 0 \tag{6}$$

The interpretation is that consumption of the numeraire *must* decrease. This makes intuitive sense: since utility is only a function of housing and numeraire consumption, if housing consumption increases with distance from the CBD then numeraire consumption must decrease to ensure equal utility. Notice that if we assume *equal* housing everywhere—as is sometimes done in simpler versions of the model—then numeraire consumption is constant and the house price adjusts exactly with commuting costs.

Problem 3: Let $S(x) = K(x)/L$ represent the capital-land ratio, derive the capital-land gradient $\frac{\partial S(x)}{\partial x}$ and give its sign.

Answer: Profit to the developers is $\Pi(x) = P(x) * H(x) - i * K - R(x) * L$. If we take the ratio of the FOC's (wrt K and L) we have:

$$S^*(x) \equiv \frac{K^*(x)}{L*} = \frac{1-\alpha}{\alpha} * \frac{R(x)}{i} \tag{7}$$

Brueckner shows generally how we can find the sign of the gradients of R and S with respect to any of the parameters. Here I just show explicitly the land-rent gradient and capital-to-land ratio gradient. If we rewrite the profit function in terms of S we have:

$$\Pi(x) = L * \left(P(x) * S^{1-\alpha} - i * S - R\right) = 0 \tag{8}$$

Then we can solve for R to find the land rent gradient.
$R^*(x) = P(x)^{\frac{1}{\alpha}} * i^{\frac{\alpha-1}{\alpha}} * (1-\alpha)^{\frac{1-\alpha}{\alpha}} * \alpha$

$$\frac{\partial R^*(x)}{\partial x} = \left[P(x)^{\frac{1-\alpha}{\alpha}} * i^{\frac{\alpha-1}{\alpha}} * (1-\alpha)^{\frac{1-\alpha}{\alpha}}\right] * \frac{\partial P(x)}{\partial x} = h(S^*(R^*)) * \frac{\partial P(x)}{\partial x} < 0 \tag{9}$$

This land rent gradient must be less than zero because $\partial P(x)/\partial x < 0$. Given that the land rent gradient is negative the capital-to-land ratio is also negative:

$$\frac{\partial S(x)}{\partial x} = \frac{1-\alpha}{i*\alpha} * \frac{\partial R(x)}{\partial x} < 0 \tag{10}$$

Problem 4: Derive the density gradient and briefly interpret: how does density change as we move away from the city and why? Note: it may help you to first derive the land rent gradient, $\frac{\partial R(x)}{\partial x}$.

Answer : The population living at x is equal to the amount of floor space $H(K^*(x), L^*)$ divided by the per person consumption of floor space at x: $q(x)$. Then dividing by the land area gives us the population density $D(x)$ at x:

$$D(x) = \frac{H(x)}{q(x) * L} = \frac{h(S)}{q(x)} = \left[\frac{1-\alpha}{\alpha} * \frac{R(x)}{i}\right]^{1-\alpha} * \frac{P(x)}{\beta(w - \tau * x)} \tag{11}$$

$$\frac{\partial D(x)}{\partial x} = \frac{\partial h(S)}{\partial S} * \frac{\partial S(x)}{\partial x} * \frac{1}{q(P)} - \frac{h(S)}{q(P)^2} * \frac{dq}{dx} < 0 \tag{12}$$

The density is decreasing for two separate reasons: 1) the capital-to-land ratio is declining with distance (because land rent declines) and 2) residents consume more housing at greater distance from the CBD. In other words, if we had a simpler undergrad-style model without flexible preferences, so that everyone consumes the same amount of housing, we would still find density declining because developers construct smaller buildings further from the center.

Problem 5, Closed City Model: To save you some tedious algebra I will give you specific parameters for this question: $\beta = 0.5$, $w = 100$, $\tau = 20$, $i = 0.1$, $\alpha = 0.75$, $R_A = 100$, $N = 1000$. Calculate the equilibrium utility $\bar{u}$, the distance to the city fringe $\bar{x}$, the land rent at the CBD $R(0)$, the housing price at the CBD $P(0)$, and the density at the CBD $D(0)$ and at the fringe $D(\bar{x})$. Note: these numbers will not work out cleanly and you will need a calculator to solve.

Answer: Mathematica is a nice program for answering this type of problem because you can easily graph functions, but any program is fine. You should find:

- $\bar{u} = 1.22507$
- $\bar{x} = 4.31566$
- $R(0) = 20100$
- $P(0) = 1665.77$
- $D(0) = 536$
- $D(\bar{x}) = 19.4835$

Problem 6, Open City Model: In this question I want you to explain the effect of a transportation cost increase on the density gradient for A) the closed city model and B) the open city model. You do not have to do any calculation, just explain what should happen and why the effect of a transportation cost differs for the two situations (closed vs open).

Answer: In the closed-city model an increase in transportation cost raises the density at the center but at some positive distance $0 < x < \bar{x}_0$ will *lower* the density. This is because the price gradient becomes even steeper and thus housing consumption can actually increase far from the CBD. However, with the open city model an increase in transportation cost simply flattens the gradient everywhere as the population decreases (leaves the city).