The Monocentric City Model

Nathan Schiff
Shanghai University of Finance and Economics

Graduate Urban Economics, Lecture 2
March 4, 2025

Student presentations: Lecture 1

1. Papers on Zipf's Law in China, including: Luckstead and Devadoss (Ec. Letters 2014), Soo (Papers in Regional Science 2014), or others (get my approval first)
2. Combes, Demurger, Li, "Migration Externalities in Chinese cities," *European Economic Review*, 2015
3. Glaeser, Lu, "Human-Capital Externalities in China", *NBER WP*, 2018. Also see https://cepr.org/voxeu/columns/human-capital-externalities-china
4. Combes, Demurger, Li, Wang, "Unequal Migration and Urbanisation Gains in China," *Journal of Development Economics*, 2020
5. Dingel, Miscio, Davis, "Cities, Lights, and Skills in Developing Economies," *Journal of Urban Economics*, 2020
6. Card, Rothstein, Yi, "Location, Location, Location," *U.S. Census Bureau Working Paper*, 2023
7. An, Qin, Wu, You, "The Local Labor Market Effects of Relaxing Internal Migration Restrictions: Evidence from China," *Journal of Labor Economics*, 2024

Student presentations: Lecture 2

1. Harari, Mariaflavia, “Cities in Bad Shape: Urban Geometry in India,” *American Economic Review*, 2020
2. Zheng, Siqi and Kahn, Matthew, “Land and residential property markets in a booming economy: New evidence from Beijing,” *Journal of Urban Economics*, 2008
3. Zhou, Zhengyi, Chen, Hong, Han, Lu, and Zhang, Anming, “The Effect of a Subway on House Prices: Evidence from Shanghai,” *Real Estate Economics*, 2021
4. Gupta, Arpit, Van Nieuwerburgh, Stijn, and Kontokosta, Constantine, “Take the Q Train: Value Capture of Public Infrastructure Projects,” *Journal of Urban Economics*, 2022
5. Liu, Crocker, Rosenthal, Stuart, and Strange, William, “The Vertical City: Rent Gradients, Spatial Structure, and Agglomeration Economies,” *Journal of Urban Economics*, 2018

Student presentations: Lecture 2, cont...

1. Rosenthal, Stuart, Strange, William, and Urrego, Joaquin, "Are City Centers Losing Their Appeal? Commercial Real Estate, Urban Spatial Structure, and COVID-19," *Journal of Urban Economics: Insight*, 2022
2. Delventhal, Matthew, Kwon, Eunjee, and Parkhomenko, Andrii, "How do cities change when we work from home?" *Journal of Urban Economics: Insight*, 2022
3. Akbar, Couture, Duranton, Storeygard, "Mobility and Congestion in Urban India," *American Econmomic Review*, 2023
4. Monte, Porcher, Rossi-Hansberg, "Remote Work and City Structure," *NBER Working Paper* 31494, 2023

Motivation for the Monocentric City Model

Introduction

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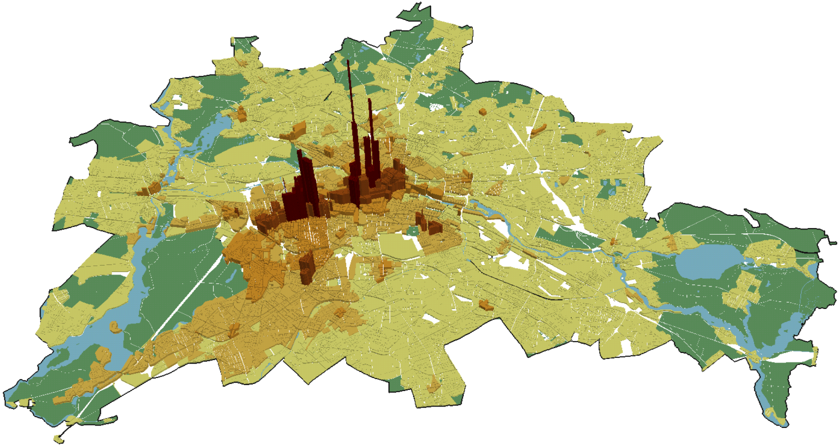
To get a better idea of these questions let's look at some maps

Source: Gilles Duranton November 7, 2015 presentation and Alain Bertaud, Feb 2002 presentation

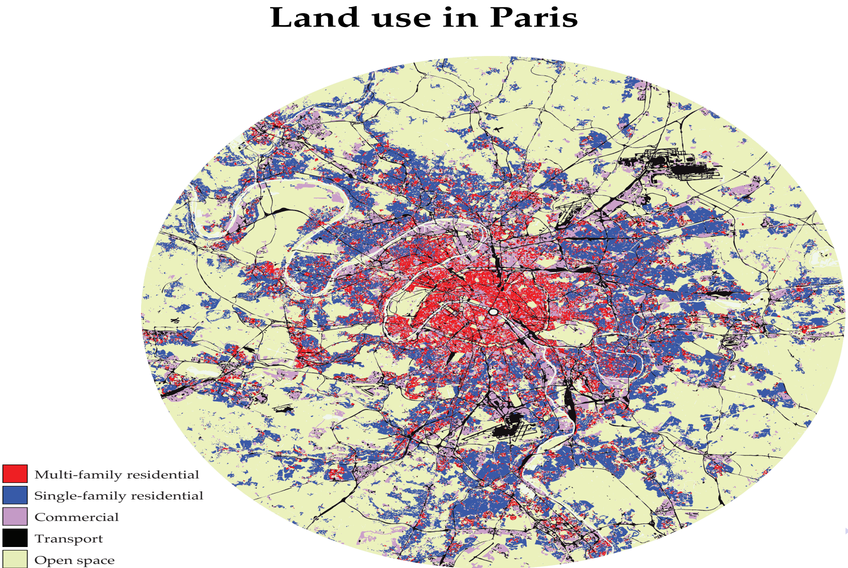
Land Price Map, Source: Duranton

Land prices in Berlin



Source: Ahlfeldt, Redding, Sturm, and Wolf (2014)

Land Use Maps, Source: Duranton



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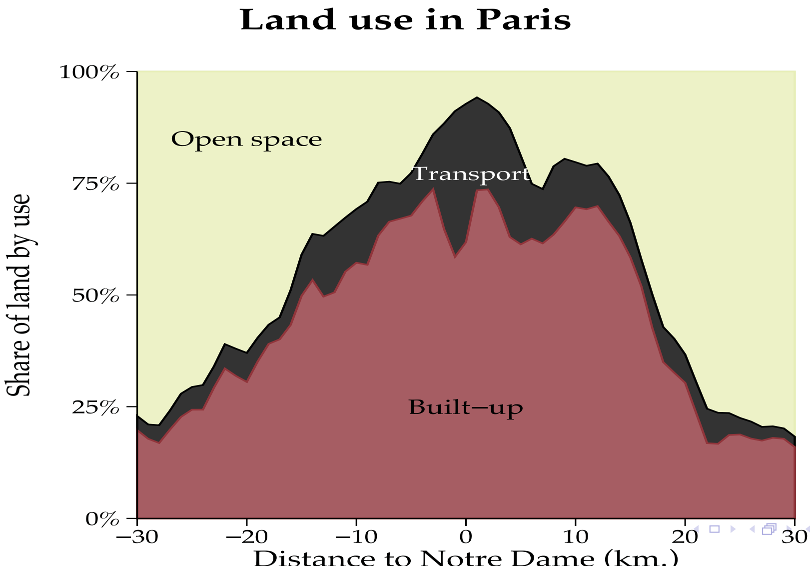

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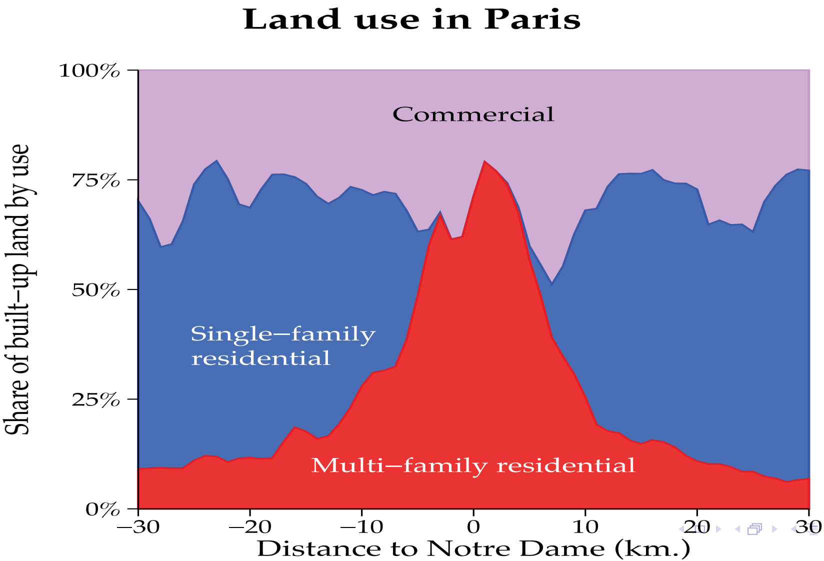
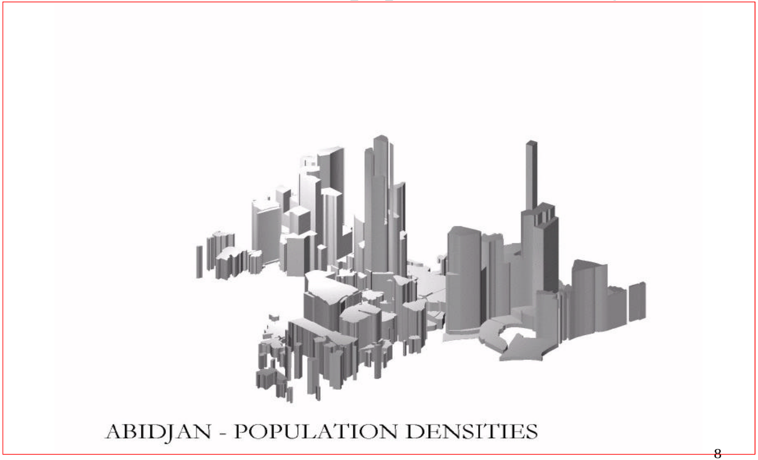


3D Density Map, Source: Bertaud

Distribution of population in Abidjan



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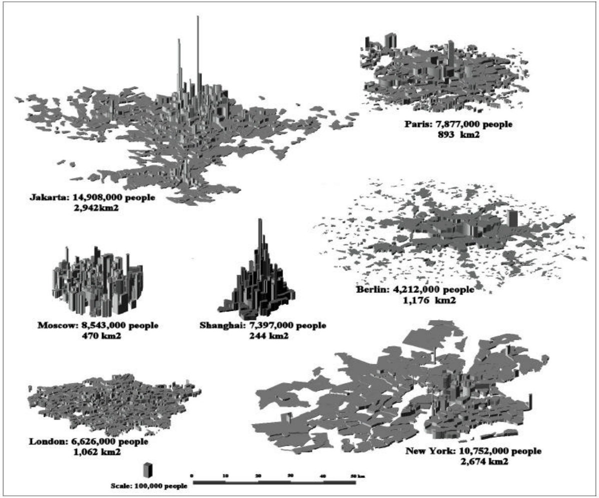
3D Density Map, Source: Bertaud

Distribution of population in Hong Kong

9

3D Density Map, Source: Bertaud

Spatial distribution of population in 7 major metropolis represented at the same scale



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Specific questions we could ask:

1. What should happen to the spatial distribution of population as Shanghai builds more subway lines extending into far districts?
2. What should happen to Shanghai residents' quality of life as transportation infrastructure improves? Does it depend on the hukou system?

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Robert Solow on the monocentric city model

Solow (Nobel 1978) on model (source: Henderson and Thisse, JUE 2024):

> *"To study the locational equilibrium of a city seems almost silly. Buildings, streets, subways, are among the most durable objects we make, and it is very expensive to move them or even to remove them. Existing patterns of location must therefore have been determined in a large part by decisions that were made and events that happened under conditions that ruled long ago. It seems far-fetched to expect that what now exists will bear much relation to what would now be an equilibrium. Nevertheless, it turns out that the equilibrium states of simple models of urban location do actuality reproduce some of the important characteristics of real cities."*

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Results:

1. Housing prices decrease with distance from the Central Business District (CBD)
2. Housing consumption increases with distance from CBD
3. Density and capital-to-land ratio decrease with distance from CBD

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In my slides I will mostly use Brueckner's notation with matching equation labels.

Modeling Residents

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Resident utility maximization problem is thus:

$$\max_q v(y - \tau * x - p(x)q(x), q) = u \tag{1}$$

Intuition from Undergrad Version

Fix housing consumption to $q(x) = \bar{q}$ for all residents, now:

$$\max_x v(y - \tau * x - p(x)\bar{q}, \bar{q}) = u \tag{1a}$$

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$\frac{\partial v}{\partial z}\left(-\tau - \frac{\partial p}{\partial x}\bar{q}\right) = 0$ or $\frac{\partial p}{\partial x} = \frac{-\tau}{\bar{q}}$

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Ex: if $p(0)$ is price at center then it must be that $y - p(0) * \bar{q} = y - \tau * x - p(x) * \bar{q}$, which is true when $p(x) = p(0) - \frac{\tau}{\bar{q}} * x$

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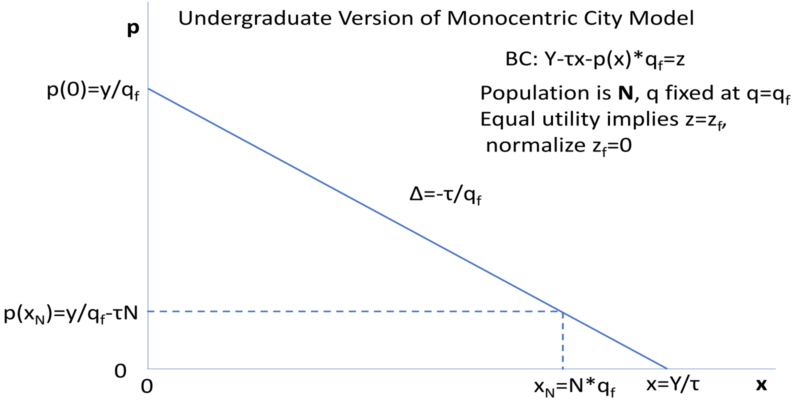
We can close basic undergrad model by assuming a fixed population N so that boundary of city is $\bar{x} = N * \bar{q}$ where price is $\bar{p}$ (note: $p(0)$ is not determined)

Undergraduate AMM: bid rent graph



Back to Brueckner: Residents' Optimal Consumption

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The equal utility condition implies:

$$v(y - \tau * x - p(x)q(x), q) = u \tag{3}$$

Deriving the Price Gradient

$$\frac{\frac{\partial v(y-\tau*x-p(x)q(x),q)}{\partial q}}{\frac{\partial v(y-\tau*x-p(x)q(x),q)}{\partial z}} = \frac{p(x)}{1} \quad (2)$$

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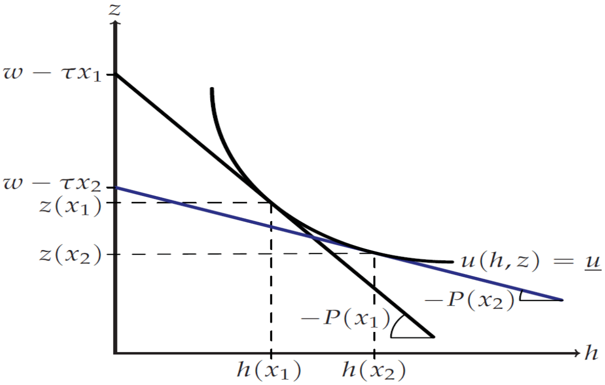
Intuition for Price Gradient: Duranton+Puga 2015



Panel (b)
Comparative statics

Consumers must be optimizing and have same utility at all x

Price Gradient: Alonso-Muth Condition

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If housing increases with dist from CBD then gradient is convex: consumers substitute cheaper housing consumption for numeraire consumption, so prices don't have to decline as quickly to compensate consumers

The Bid-Rent Function

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Bid-rent:

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Plugging in budget constraint gives:

$$\Psi(x,u) = \max_{q(x),z(x)} \left\{ \frac{y - \tau * x - z(x)}{q(x)} | v(q,z) = u \right\}$$

Alternative Derivation of Alonso-Muth

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A key condition is that utility is always equal to u, therefore we can rewrite above equation using Hicksian demand functions from expenditure minimization problem:

$$\min_{q,z} z + p(x) * q(x), \text{s.t. } v(z,q) = u \tag{1b}$$

Now: $\Psi(x,u) = \max_{q(x)} \left\{ \frac{y - \tau * x - z(q(x),u)}{q(x)} \right\}$

Alternative Derivation of Alonso-Muth

$\Psi(x,u) = \max_{q(x),z(x)} \left\{ \frac{y-\tau*x-z(x)}{q(x)} | v(q,z) = u \right\}$

A key condition is that utility is always equal to u, therefore we can rewrite above equation using Hicksian demand functions from expenditure minimization problem:

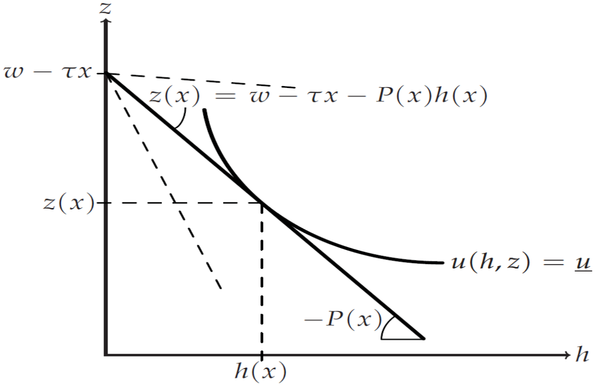
$$\min_{q,z} z + p(x) * q(x), \text{s.t. } v(z,q) = u \tag{1b}$$

Now: $\Psi(x,u) = \max_{q(x)} \left\{ \frac{y-\tau*x-z(q(x),u)}{q(x)} \right\}$

Then, taking derivative wrt x and using the envelope theorem:

$\frac{dp(x)}{dx} = \frac{d\Psi(x,u)}{dx} = -\frac{\tau}{q(x)}$

Intuition for Bid-Rent: Duranton+Puga 2015



Panel (a)
Deriving housing prices in x

Note: budget constraint pivots around z-intercept (not a shift, as when deriving expenditure function)

Housing Consumption Gradient

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Gradient of Hicksian housing demand is:

$\frac{dq(p,u,x)}{dx} = \frac{\partial q}{\partial p} * \frac{\partial p}{\partial x} > 0$

Therefore, we know that housing consumption is *increasing* with distance; the housing price is cheaper so consumers substitute towards housing

Housing Production and Developers

Housing Production

The housing construction industry is perfectly competitive with a *concave* CRS production function

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Profit: $\Pi(x) = L * (p(x) * h(S) - i * S - r(x))$

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Totally differentiating these conditions will allow us to derive the land-rent gradient and capital-to-land ratio gradient

Deriving land rent and capital-to-land gradient

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$$\frac{\partial p}{\partial \phi} * \frac{\partial h}{\partial S} + p * \frac{\partial^2 h}{(\partial S)^2} * \frac{\partial S}{\partial \phi} = 0 \tag{13}$$

$$(p * \frac{\partial h}{\partial S} - i) * \frac{\partial S}{\partial \phi} + \frac{\partial p}{\partial \phi} h = \frac{\partial r}{\partial \phi} \tag{14}$$

Land rent and capital-land gradient

Finally, by inserting the FOC (11) into (14) we get:

$$\frac{\partial r}{\partial \phi} = h * \frac{\partial p}{\partial \phi} \tag{15}$$

Re-arranging (13) gives:

$$\frac{\partial S}{\partial \phi} = -\frac{\partial h}{\partial S} * (p * \frac{\partial^2 h}{(\partial S)^2})^{-1} * \frac{\partial p}{\partial \phi} \tag{16}$$

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This gives us:

$$\frac{\partial r}{\partial x} < 0, \text{ and }, \frac{\partial S}{\partial x} < 0 \tag{17}$$

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Density decreases for *two* reasons: 1) capital-to-land ratio declines 2) per-person housing consumption increases

Summary of Results

$$\frac{\partial p}{\partial x} = \frac{-\tau}{q(x)} < 0 \tag{r1}$$

$$\frac{dq}{dx} = \frac{\partial q(p,u)}{\partial p} * \frac{\partial p}{\partial x} > 0 \tag{r2}$$

$$\frac{\partial r}{\partial x} = h(S) * \frac{\partial p}{\partial x} < 0 \tag{r3}$$

$$\frac{\partial S}{\partial x} = -\frac{\partial h}{\partial S} * (p * \frac{\partial^2 h}{(\partial S)^2})^{-1} * \frac{\partial p}{\partial x} < 0 \tag{r4}$$

$$\frac{\partial D(x)}{\partial x} = \frac{\partial h(S)}{\partial S} * \frac{\partial S(x)}{\partial x} * \frac{1}{q(x)} - \frac{h(S)}{q(x)^2} * \frac{dq}{dx} < 0 \tag{r5}$$

Equilibrium and Comparative Statics

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- utility $\bar{u}$ is exogenous; need to solve for endogenous population N and fringe $\bar{x}$

I'll discuss how to close the model and the intuition, see proofs in Brueckner

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$$r(\bar{x}, y, \tau, u) = r_A \tag{18}$$

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Note: equation 19 is simpler than in Brueckner due to above assumptions

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Note: above only holds for linear city; see this Appendix slide for solving generally

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We know price at fringe must be equal to construction cost at fringe, can invert to find $\bar{x}$:

$$p(\bar{x}) = C(i, r(\bar{x})) = C(i, r_A)$$

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We are interested in how changes in the main parameters (r_A, y, τ, N) affect housing prices $p(x)$, housing consumption $q(x)$, land rent $r(x)$, and the capital to land ratio $S(x)$

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Thus r_A and N only affect $p(x), q(x), r(x), S(x)$ *indirectly* by changing u; y and τ will affect these variables both directly and indirectly through u

Closed City: Increase in Agricultural Rent

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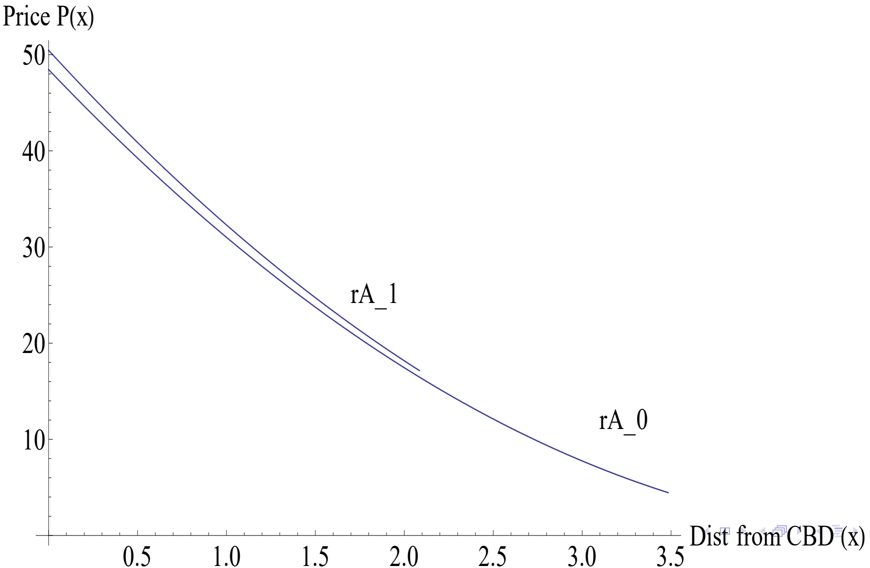
Q: Why can't the fringe $\bar{x}$ stay the same?

Example: Closed City, Agricultural Rent Increase



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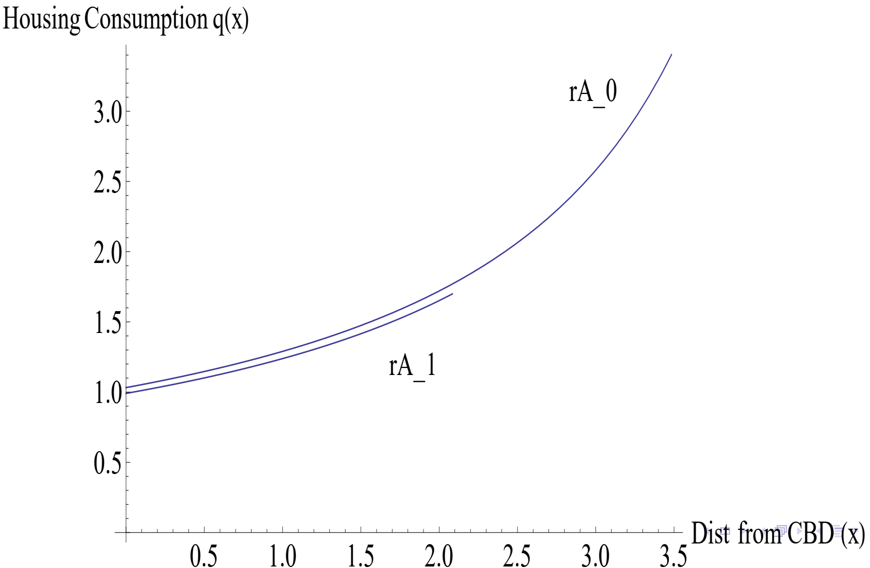

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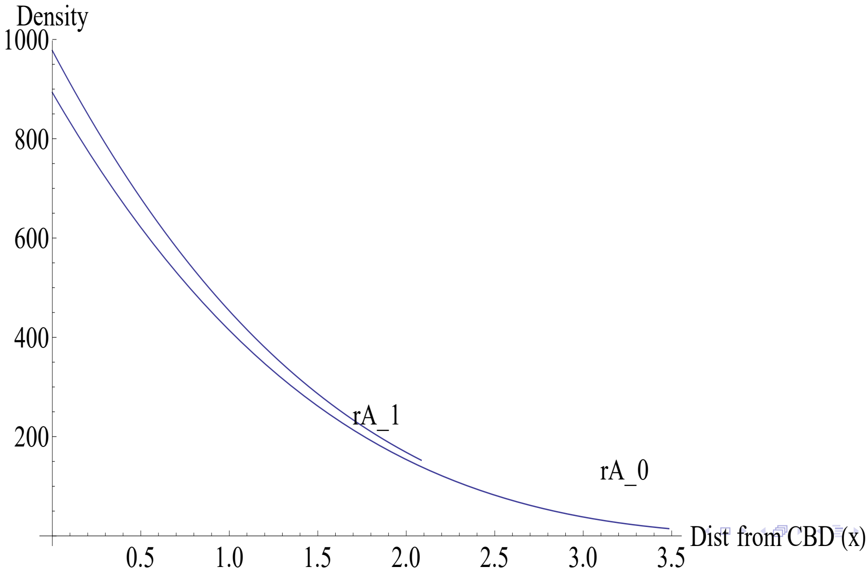

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City expands geographically but not enough so that density at x is constant

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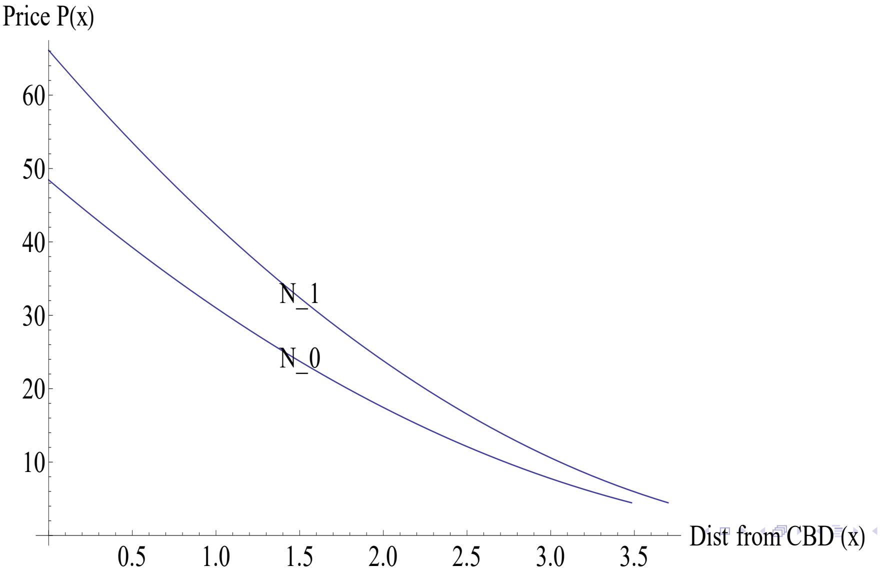
2. Fringe expands $\bar{x}_1 > \bar{x}_0$
3. Price gradient *shifts up* and steeper
4. Density rises everywhere–why not just for $x > \bar{x}_0$?

City expands geographically but not enough so that density at x is constant

Intuition (example): new residents locating outside of fringe would have lower utility than at fringe, thus bid-up prices. Increase in price makes more central locations desirable, all locations will increase. Generally, more room to increase price closer to center because housing consumption is smaller, but also depends on functional form assumptions (ex: how convex is cost of building higher)

Example: Closed City, Population Increase



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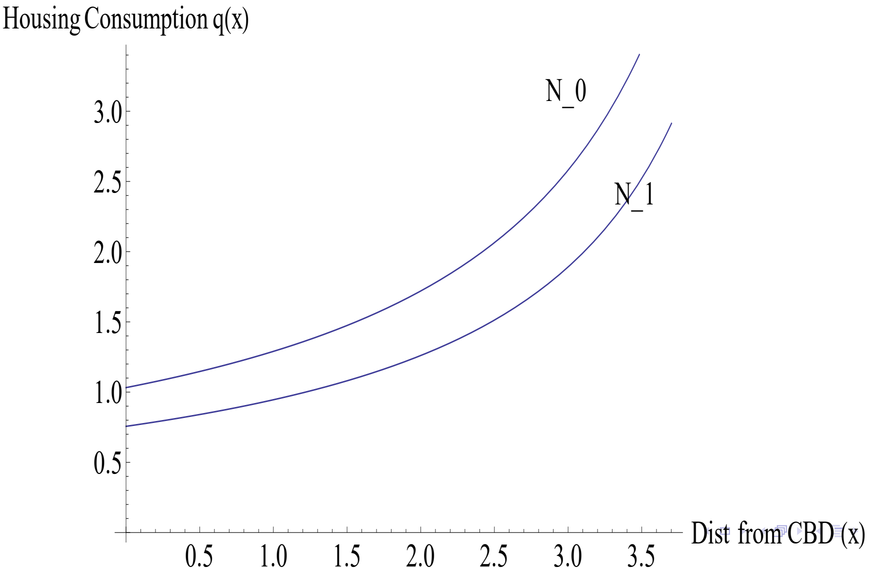

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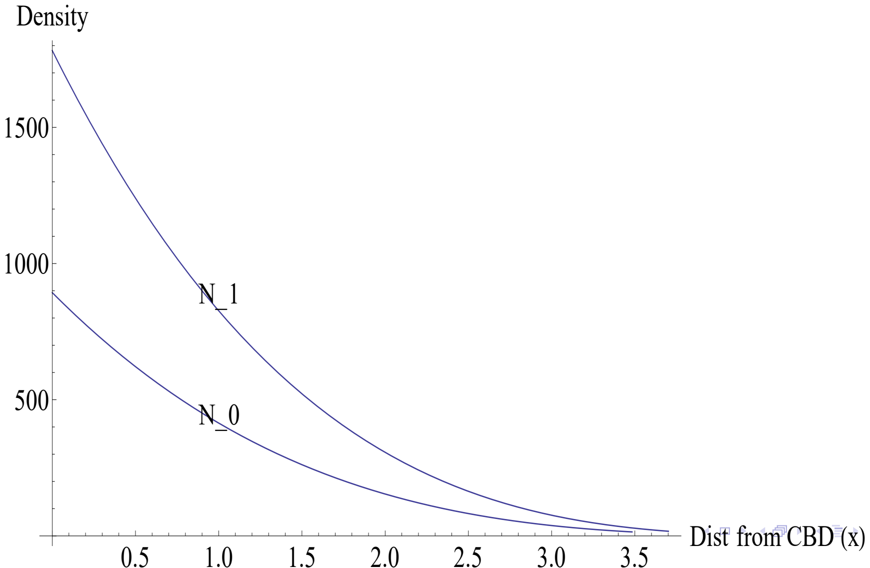

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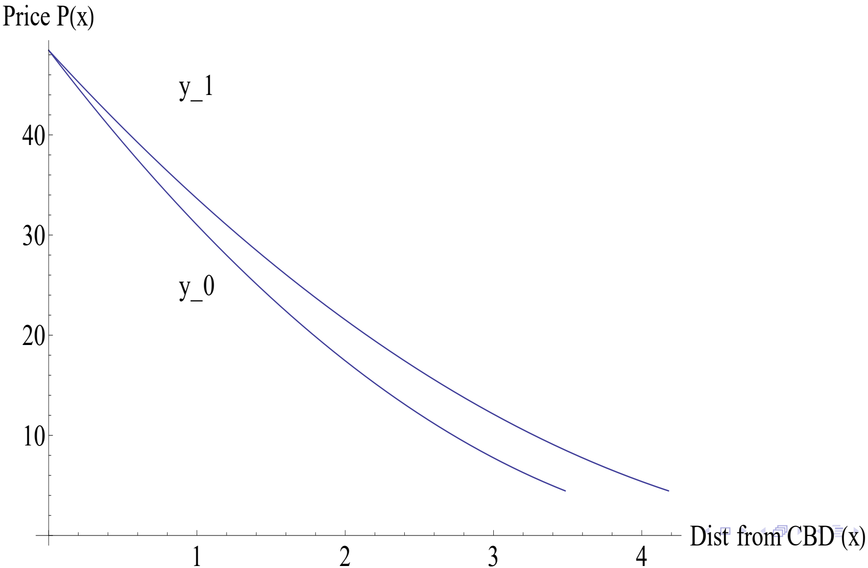
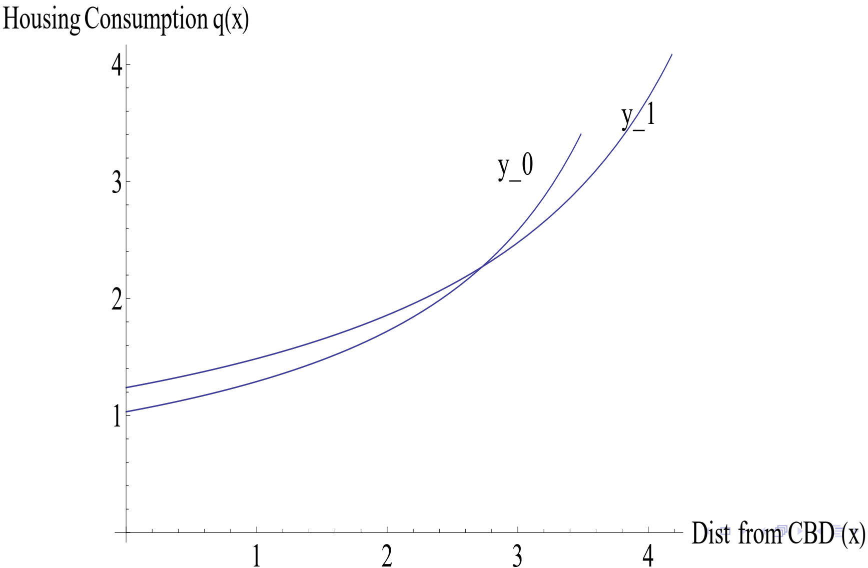
5. *For this functional form* housing consumption gradient also rotates; tradeoff between housing and numeraire consumption (technical detail)

City expands geographically, most people consume more housing, live further away, increases density away from CBD

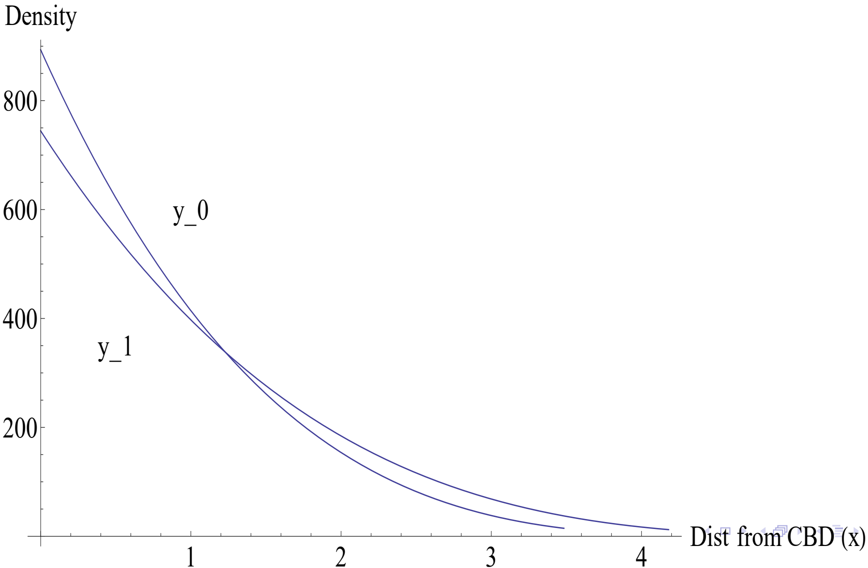
Closed City, Income Increase, Price Gradient



Closed City, Income Increase, Housing Gradient



Closed City, Income Increase, Density Gradient



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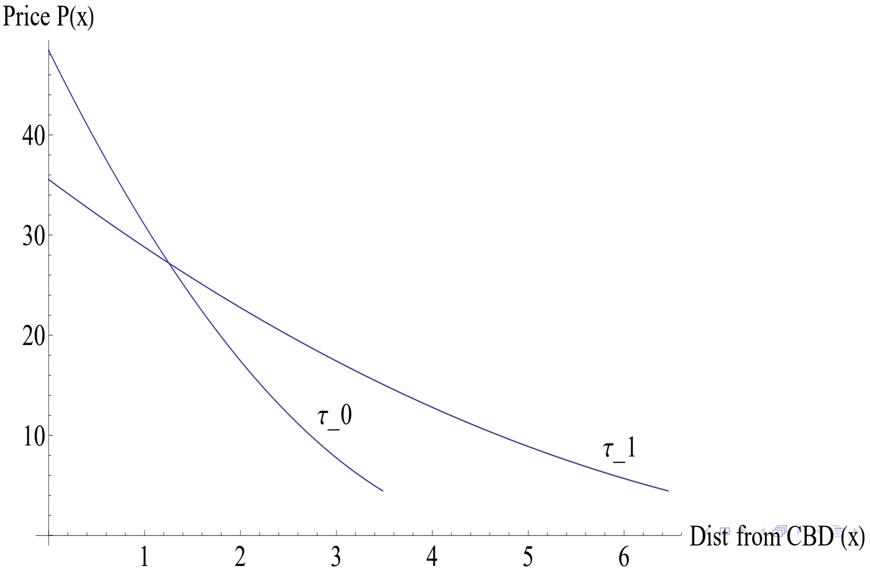
Basically more distant locations become more attractive, decreasing demand for central locations

Example: Closed City, Transportation Cost Decrease



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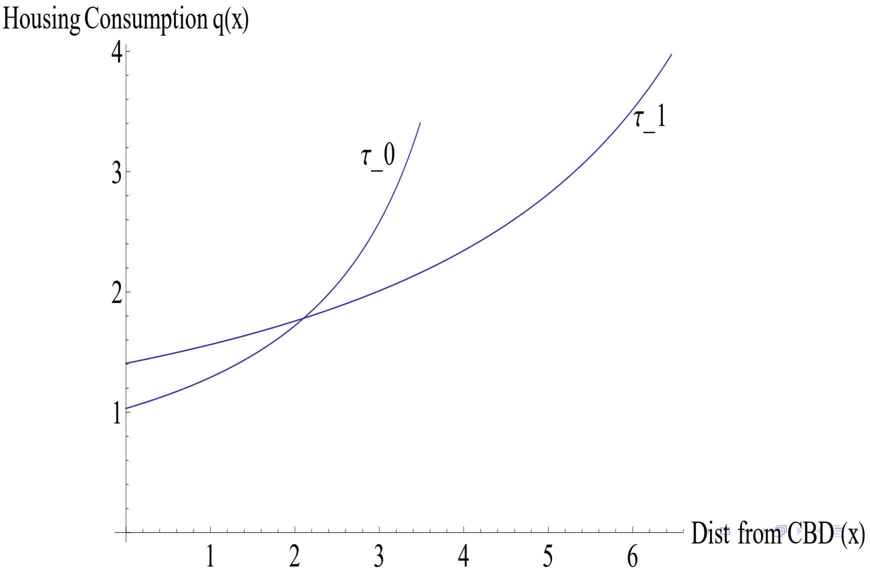

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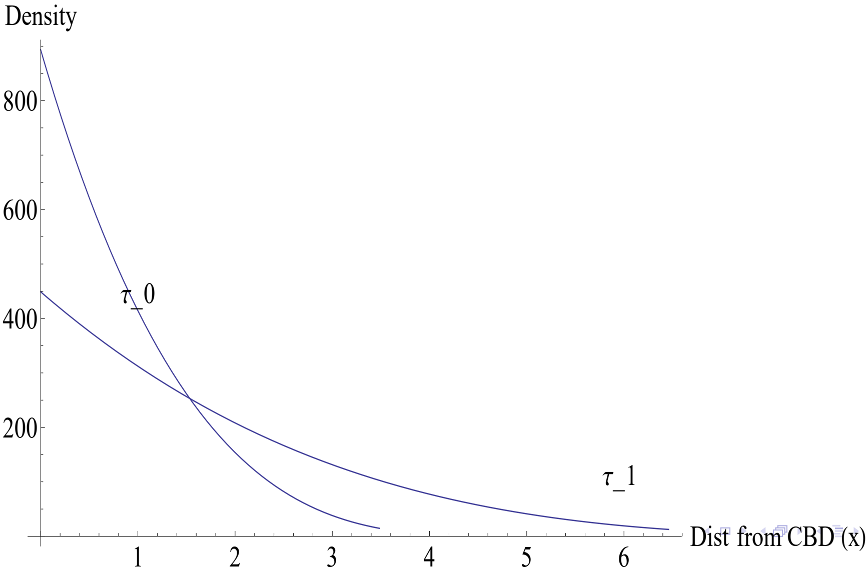

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Then, population flows in or out to restore original utility level, with resulting closed-city effect of population change (long-run)

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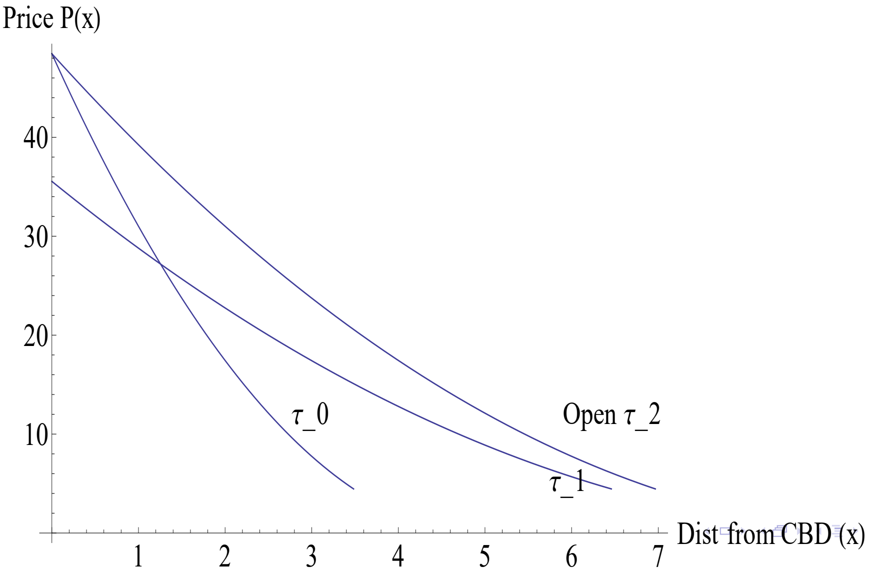

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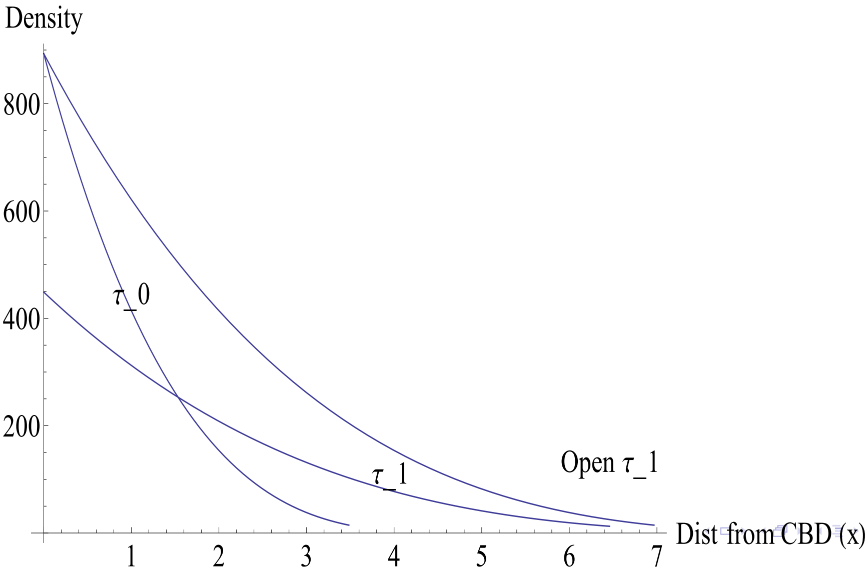

Empirical Gradients

Some empirical gradients

The following are density estimates from various cities collected by Bertaud and Malpezzi.

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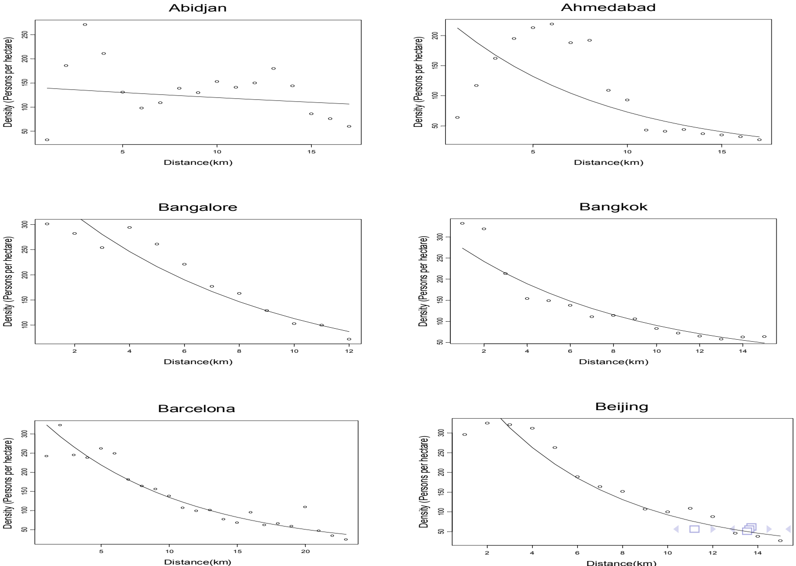
The data is not public, CBD definition is subjective,year is unclear; still quite informative

Source: Bertaud and Malpezzi, “The Spatial Distribution of Population in 48 World Cities: Implications for Economies in Transition”, World Bank Report 2003

Empirical Density Gradients



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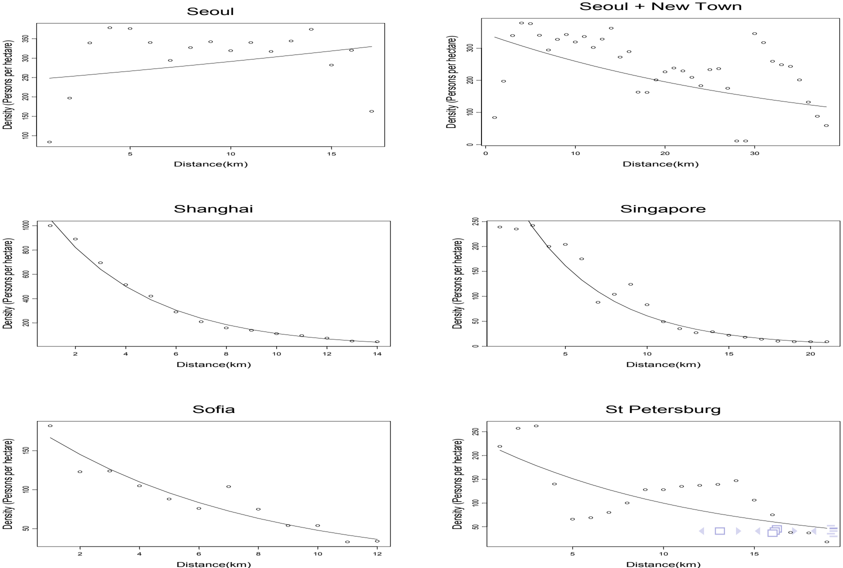

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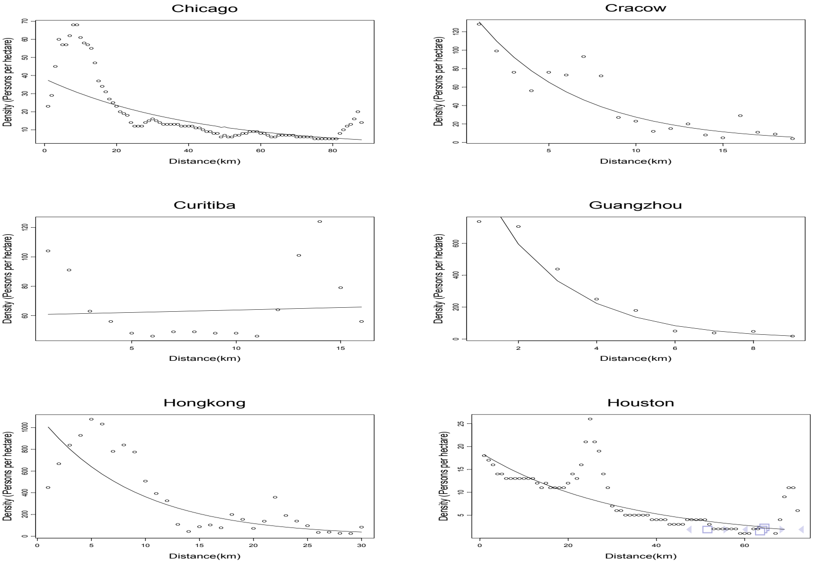

Final thoughts:extensions and weaknesses

Extensions

Transport cost: we assumed a simple monetary cost but a common alternative is a time cost (decreases working hours). Rappaport (2014) adds leisure so that commuting decreases leisure time, consistent with empirical evidence of strong disutility of commuting.

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Model is useful for thinking about general patterns, but not suited to structural estimation; modern empirical approaches use "quantitative spatial models"

Concluding: Weaknesses of the Model

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Next Class: Read Jerch et. al. article; LeRoy and Sonstelie may help with theoretical intuition (both articles on my website)

References for this Lecture

This lecture is based on the following references:

1. Brueckner,Jan K., *Handbook of Regional and Urban Economics*, Volume 2, Ch. 20, 1987
2. Fujita, Masahisa, *Urban Economic Theory*, Ch. 2-3, 1989
3. Duranton, Gilles and Puga, Diego, *Handbook of Economic Growth*, Volume 2B, Ch. 5, 2014
4. Duranton, Gilles and Puga, Diego, *Handbook of Regional and Urban Economics*, Volume 5, 2015
5. Duranton, Gilles, *Empirics of housing, land use, and location choice*, Presentation given at Frontiers of Urban Economic Conference, November 7, 2015

Solving the model generally

We have two unknowns: 1) equilibrium utility $\bar{u}$ and 2) the distance to the fringe, $\bar{x}$.

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$$r(\bar{x}, y, \tau, \bar{u}) = r_A \tag{18}$$

$$\int_0^{\bar{x}} D(x)dx = \int_0^{\bar{x}} h(S(x))/q(x)dx = N \tag{19}$$

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Notice that both $S(x)$ and $q(x)$ depend on the housing price, $p(x)$, where $S(x)$ is related to $p(x)$ through $r(x)$

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Then both equations can be solved simultaneously for $\bar{u}$ and r_A:

$$r(\bar{x}, y, \tau, \bar{u}) = r_A \tag{18}$$

$$\int_0^{\bar{x}} h(S(u, x))/q(u, x)dx = N \tag{19b}$$